

BLOCKWISE SIMPLIFICATION OF THE KERR–NEWMAN BLACK HOLE’S THERMODYNAMICAL STATE SPACE RUPPEINER GEOMETRY

Edward Anderson*

APC AstroParticule et Cosmologie, Université Paris Diderot CNRS/IN2P3, CEA/Irfu, Observatoire de Paris, Sorbonne Paris Cité, 10 rue Alice Domon et Léonie Duquet, 75205 Paris Cedex 13, France.

Abstract

A coordinate system that blockwise-simplifies the Kerr–Newman black hole’s thermodynamical state space Ruppeiner metric is constructed, with discussion of the limiting cases corresponding to simpler black holes. It is deduced that one of the three conformal Killing vectors of the Reissner–Nordström and Kerr cases (whose thermodynamical state space metrics are 2 by 2 and, moreover, conformally flat) survives generalization to the Kerr–Newman case’s 3 by 3 thermodynamical state space metric.

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* eanderso@apc.univ-paris7.fr

1 Introduction

This brief paper concerns the Riemannian geometry of thermodynamical state spaces, which is of wider interest as one possible means of studying phase transitions. I follow Ruppeiner’s approach [1, 2, 3]; see e.g. [4, 5, 6, 7] for other approaches. I consider the black holes case of this, for which the thermodynamical fluctuation formalism itself has difficulties concerning stability about a maximum in the total entropy, so that an alternative to it is desirable.

Ruppeiner [1, 3] considers the expansion

$$\Delta S_{\text{Tot}} = F_{\mu} \Delta X^{\mu} + F_{e\mu} \Delta X_e^{\mu} + \frac{1}{2} \frac{\partial F_{\mu}}{\partial X_e^{\nu}} \Delta X^{\mu} \Delta X^{\nu} + \frac{1}{2} \frac{\partial F_{e\mu}}{\partial X_e^{\nu}} \Delta X_e^{\mu} \Delta X_e^{\nu} + \dots \quad (1)$$

Here, $S_{\text{Tot}} = S_{\text{BH}} + S_e$ is the total entropy of the universe, the standard black hole thermodynamical variables are denoted by X^{μ} [e.g. $(X^1, X^2, X^3) := (M, J, Q)$ for the Kerr–Newman black hole], S is entropy, and

$$F_{\mu} = \partial S / \partial X^{\mu} \quad (2)$$

The subscript ‘e’ stands for ‘equilibrium’. Then for a very large, extensive environment,

$$\Delta S_{\text{Tot}} = -\frac{1}{2} g_{\mu\nu} \Delta X^{\mu} \Delta X^{\nu} \quad (3)$$

transforms as a scalar since this ΔS depends only on the initial and final thermodynamical states. One can view the corresponding Ruppeiner geometry as an *information geometry of Fisher type* [8]. The associated Ricci scalar is then argued by Ruppeiner to be a measure of interaction strength. Ruppeiner computed the form of this for the Kerr–Newman black hole solution’s thermodynamical state space in e.g. [3].

Further coverage of the Kerr–Newman black hole case is the subject of this paper. This is a case of particular importance due to this 4- d black hole [9] likely being astrophysically realized (at least to very good approximation as a model and for very small values of the charge parameter). This is due to its relevant and not excessively high symmetry, uniqueness theorems leading to it, and its capacity to arise as the end-product of the evolution of somewhat more irregular configurations. See e.g. [10, 6] for previous work on the geometry of thermodynamic state spaces for 4- d black holes. In the present paper, a superior coordinate system for the Ruppeiner thermodynamical state space geometry for the Kerr–Newman black hole is found (Sec 3); it is superior from the point of view of block-minimality. This uses Davies coordinates [11], mass-homogenization and then a new coordinate transformation; I also present the overall composite transformation. I discuss the value of block-minimality and present limiting cases (Reissner–Nordström, Kerr, extremal) in the Conclusion (Sec 4), alongside showing using my coordinate system that 1 of the 3 conformal Killing vectors of the limiting cases’ thermodynamical state spaces survives the generalization to the Kerr–Newman case.

2 Ruppeiner thermodynamical geometry in the Kerr–Newman case

For the physically-valuable 4- d Kerr–Newman solution, the entropy is given by [12]

$$S = \pi K_B [2M^2(1 + \bar{E}) - Q^2] \quad (4)$$

for

$$\text{‘nonextremality’ } \bar{E} := \sqrt{1 - Q^2/M^2 - J^2/M^4} \quad (5)$$

With these standard thermodynamic state variables as coordinates, the thermodynamical Ruppeiner metric is as in Fig 1a). The idea is then to blockwise-simplify this by applying coordinate transformations.

3 Its blockwise simplification

I first use a transformation well-known to Ruppeiner [3] and which goes at least as far back as to Davies [11]:

$$q := Q/M, \quad j := J/M^2; \quad (6)$$

these are a particular case of dimensionless variables. I note that these then simplify the blockwise form of the thermodynamical Ruppeiner metric as in Fig 1b). I next apply a mass-homogenizing/ M -dependence-freeing transformation

$$\mathcal{M} := \ln M, \quad (7)$$

so as to have a fully homogenized version of Davies-type coordinates. This casts the metric in the form of Fig 1c).

Finally, I obtain a further cross-term removing coordinate transformation alongside a simplification for the isolated 1×1 block by solving a pair of differential equations, yielding the coordinate transformation

$$\mathcal{F} := \arcsin(j/\sqrt{1-q^2}) \quad (8)$$

by which the metric is cast in the final blockwise-simplified form of Fig 1d). [I use $\mathcal{Q} := q$ to have the same typeface for my final set of three blockwise-simplifying coordinates.]

Composing, the complete transformation is

$$\mathcal{M} = \ln M, \quad \mathcal{Q} = Q/M, \quad \mathcal{F} = \arcsin(J/M\sqrt{M^2 - Q^2}), \quad (9)$$

with inverse

$$M = \exp \mathcal{M}, \quad Q = \mathcal{Q} \exp \mathcal{M}, \quad J = \sqrt{1 - \mathcal{Q}^2} \exp(2\mathcal{M}) \sin \mathcal{F}. \quad (10)$$

The form of \bar{E} progresses as follows along the chain of coordinate transformations:

$$\bar{E} = \sqrt{1 - Q^2/M^2 - J^2/M^4} = \sqrt{1 - q^2 - j^2} = \sqrt{1 - \mathcal{Q}^2} \cos \mathcal{F}. \quad (11)$$

a)

$$g = \pi k_B \begin{pmatrix} 4(1 + \bar{E}) + \frac{2}{\bar{E}} \left(\frac{Q^2}{M^2} - \frac{2J^2}{M^4} \right) - \frac{2}{\bar{E}^3} \left(\frac{Q^2}{M^2} + \frac{2J^2}{M^4} \right)^2 & \frac{2Q}{M\bar{E}^3} \left(\frac{Q^2}{M^2} + \frac{2J^2}{M^4} \right) & \frac{2J}{M\bar{E}^3} \left(2 - \frac{Q^2}{M^2} \right) \\ \bullet & -2 \left[\frac{1}{\bar{E}^3} \left(1 - \frac{J^2}{M^4} \right) + 1 \right] & \frac{-2JQ}{M^4 \bar{E}^3} \\ \bullet & \bullet & \frac{-1}{M^2 \bar{E}^3} \left(1 - \frac{Q^2}{M^2} \right) \end{pmatrix}$$

$M \qquad \qquad \qquad Q \qquad \qquad \qquad J$

b)

$$g = \pi k_B \begin{pmatrix} 4 \left(1 - \frac{q^2}{2} + \frac{1 - q^2}{\bar{E}} \right) & -2Mq \left(1 + \frac{1}{\bar{E}} \right) & 0 \\ \bullet & -2M^2 \left(\frac{1 - j^2}{\bar{E}^3} - 1 \right) & \frac{-2M^2 j q}{\bar{E}^3} \\ \bullet & \bullet & -2M^2 \frac{1 - q^2}{\bar{E}^3} \end{pmatrix}$$

$M \qquad \qquad \qquad q \qquad \qquad \qquad j$

c)

$$g = \frac{2\pi k_B e^{2m}}{\bar{E}} \begin{pmatrix} 2(1 - q^2) + (2 - q^2)\bar{E} & -(1 + R)q & 0 \\ \bullet & \bar{E} + (j^2 - 1)/\bar{E}^2 & -qj/\bar{E}^2 \\ \bullet & \bullet & (q^2 - 1)/\bar{E}^2 \end{pmatrix}$$

$m \qquad \qquad \qquad q \qquad \qquad \qquad f$

d)

$$g = \frac{2\pi k_B e^{2m}}{\bar{E}} \begin{pmatrix} 2(1 - \mathcal{Q}^2) + (2 - \mathcal{Q}^2)\bar{E} & -\mathcal{Q}(1 + \bar{E}) & 0 \\ \bullet & \bar{E} - 1/(1 - \mathcal{Q}^2) & 0 \\ \bullet & \bullet & -1 \end{pmatrix}$$

$m \qquad \qquad \qquad \mathcal{Q} \qquad \qquad \qquad \mathcal{F}$

Figure 1: The Ruppeiner thermodynamic metric for the Kerr–Newman black hole in a) standard state variable coordinates, b) Davies coordinates, c) my mass-homogenized version of Davies coordinates and d) my block-minimal coordinates.

4 Discussion

- 1) Limiting cases of physical interest**
- i) $\mathcal{F} = 0$ corresponds to $J = 0$, i.e. the Reissner–Nordström black hole. Here it is advantageous to, rather, work with the Weinhold geometry, which can be cast in a manifestly flat form by a different chain of coordinate transformations [6].
 - ii) Holding q constant gives a 2×2 diagonal block (here, Davies coordinates suffice to diagonalize). This includes Kerr for $q = 0$, c.f. [3], and is also conformally flat.
 - iii) The extremal values $\pm\pi/2$ of \mathcal{F} (see Fig 2) correspond to $j^2 + q^2 = 1$, i.e. the extremal Kerr–Newman black hole. Here, this paper’s coordinate system breaks down; moreover the thermodynamic state space in this case comes out straightforwardly in $x = 2M$, $t = \sqrt{2}Q$ coordinates as flat (2×2 Minkowski metric), so this case is easily elsewhere covered.

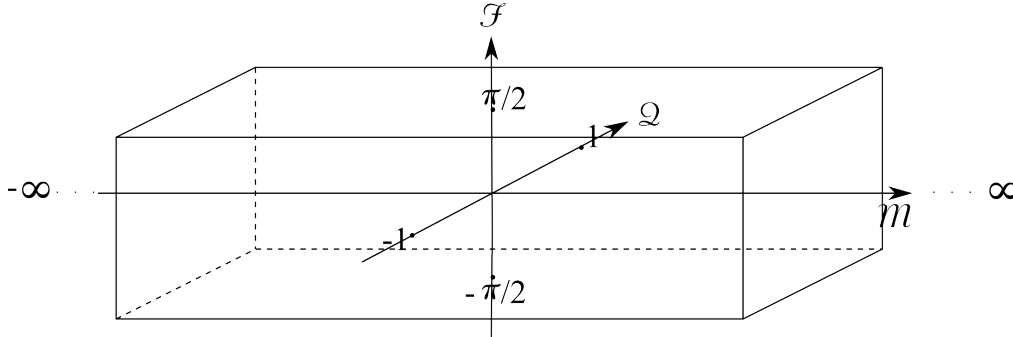


Figure 2: Shape of the Kerr–Newman thermodynamical state space in this paper’s coordinates. The extremal values of \mathcal{F} are not covered by these coordinates.

2) Note on the value of block-minimality. This is a weakening of the concept of diagonality for cases in which diagonality itself cannot be attained; this paper’s working does not look to extend to a removal of the last independent off-diagonal term. Block-minimality is likely useful in helping to further characterize this paper’s 3-geometry that arises in a physically interesting context. As an example of the usefulness of block-minimality, Gibbons–Pope coordinates for \mathbb{CP}^2 [13] subsequently serve as useful cyclic coordinates for dynamics on \mathbb{CP}^2 , in the analysis of conserved quantities on this space and in separating the Schrödinger equation on this space [14].

3) Conformal Killing vectors (CKV’s) Each of the Reissner–Nordström and Kerr black hole thermodynamic state spaces have 3 CKV’s by conformal flatness. In the generalization to Kerr–Newman black hole thermodynamical state space, at least one CKV survives: $\partial/\partial\mathcal{M}$. The homogenizing transformation that unveiled this here does have a counterpart in the case of the simpler Kerr black hole thermodynamical state space itself, though this kind of transformation was not picked up upon there, probably because of the demonstration in this case of conformal flatness, trivializing the search for CKV’s.

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References

- [1] G. Ruppeiner, Phys. Rev. A. **20** 1608 (1979); **24** 488 (1981).
- [2] G. Ruppeiner, Rev. Mod. Phys. **67** 605 (1995); Erratum-ibid. **68** 313 (1996).
- [3] G. Ruppeiner, Phys. Rev. **D78** 024016 (2008), arXiv:0802.1326.
- [4] F. Weinhold, J. Chem. Phys. **63**, 2479; 2484; 2488; 2496 (1975); **65** 559 (1976); Acc. Chem. Res. **9** 236 (1976).
- [5] S. Ferrara, G.W. Gibbons and R. Kallosh, Nucl. Phys. **B500** 75 (1997).
- [6] J.E. Åman, I. Bengtsson and N. Pidokrajt, Gen. Rel. Grav. **35** 1733 (2003), gr-qc/0304015; **38** 1305 (2006) gr-qc/0601119; J.E. Åman and N. Pidokrajt, Phys. Rev. **D73** 024017 (2006), arXiv:0510139.
- [7] J.L. Álvarez, H. Quevedo and A. Sánchez, Phys. Rev. **D77** 084004 (2008), arXiv:0801.2279; H. Quevedo and A. Vázquez, AIP Conf. Proc. **977** 165 (2008), arXiv:0712.0868.
- [8] R.A. Fisher, Phil. Trans. R. Soc. Lond. **A222** 309 (1922).
- [9] E.T. Newman et al, J. Math Phys. **6** 918 (1965); B. Carter, in *Black Holes (Les Houches Lectures)* ed. B. DeWitt and C. DeWitt (Gordon and Breach, New York 1972); R.M. Wald, *General Relativity* (University of Chicago Press, Chicago 1984).

- [10] D.A. Johnston, W. Janke and R. Kenna, *Acta Phys. Polon.* **B34** 4923 (2003), [cond-mat/0308316](#); J.-Y. Shen, R.-G. Cai, B. Wang, and R.-K. Su, *Int. J. Mod. Phys.* **A22** 11 (2007), [gr-qc/0512035](#); B. Mirza and M. Zamani-Nasab, *JHEP* 0706:059 (2007), [arXiv:0706.3450](#); J.E. Āman and N. Pidokrajt, [arXiv:0801.0016](#). A.J.M. Medved, *Mod. Phys. Lett.* **A23** 2149 (2008), [arXiv:0801.3497](#); S.-W. Wei, Y.-X. Liu, C.-E. Fu and H.-T. Li, [arXiv:0911.0270](#); J.E. Āman, N. Pidokrajt and J. Ward, *AIP Conf. Proc.* **1122** 181 (2009), [arXiv:0901.2944](#) A. Sahay, T. Sarkar, and G. Sengupta, *JHEP* 1004:118 (2010), [arXiv:1002.2538](#); 1007:082 (2010), [arXiv:1004.1625](#); R. Banerjee, S.K. Modak and S. Samanta, *Phys. Rev.* **D84** 064024 (2011), [arXiv:1005.4832](#); R. Banerjee, S. Ghosh and D. Roychowdhury, *Phys. Lett.* **B696** 156 (2011), [arXiv:1008.2644](#); H. Liu, H. Lu, M. Luo and K.-N. Shao, *JHEP* 1012:054 (2010), [arXiv:1008.4482](#); S. Bellucci and B.N. Tiwari, *JHEP* 1011:030, (2010) [arXiv:1009.0633](#).
- [11] P.C.W. Davies, *Proc. Roy. Soc. Lond. A.* **353** 499 (1977).
- [12] L. Smarr, *Phys. Rev. Lett.* **30** 71 (1973).
- [13] G.W. Gibbons and C.N. Pope, *Commun. Math. Phys.* **61** 239 (1978); C.N. Pope, *Phys. Lett.* **97B** 417 (1980).
- [14] A.J. MacFarlane, *J. Phys. A: Math. Gen.* **36** 7049 (2003); E. Anderson, [arXiv:1202.4186](#); [arXiv:1202.4187](#); [arXiv:1111.1472](#); E. Anderson and S.A.R. Kneller, [arXiv:1303.5645](#).